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On uniqueness of characteristic classes

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ABSTRACT

We give an axiomatic characterization of maps from algebraic K-theory. The results apply to a large class of maps from algebraic K-theory to any suitable cohomology theory or to algebraic K-theory. In particular, we obtain comparison theorems for the Chern character and Chern classes and for the Adams operations and λ-operations on higher algebraic K-theory. We show that the Adams operations and λ-operations defined by Grayson agree with the ones defined by Gillet and Soulé.

0. Introduction

In this paper we address the problem of comparing maps from the algebraic K-groups of a scheme to either algebraic K-groups or suitable cohomology theories. This type of question often arises when one constructs a map that is supposed to induce, in some particular cohomology theory, a specific regulator or known map, and one needs to show that the map is indeed the expected one. Examples of this situation are found in the construction of the Beilinson regulator given by Burgos and Wang in [6], in the regulator defined by Goncharov in [13] and in the definition of the Adams operations given by Grayson in [14].

The various natures of the constructions usually mean that direct comparison is not an available option and one is forced to turn to theoretical tricks. In this work, we identify sufficient conditions for two maps to agree, thus obtaining an axiomatic characterization of maps from K-theory. These aim to extend known characterization theorems for characteristic classes at K_0, such as the characterization of the Chern classes [17, Th1] and the splitting principle for Adams operations [1].

As a main consequence, we give a characterization of the *Adams operations* and λ-*operations* on higher algebraic K-theory and of the *Chern character* and *Chern classes* on a suitable cohomology theory (see Sections 4.2, 4.3 and 5.3).

In particular, we show that the Adams operations defined by Grayson in [14] agree with the ones defined by Gillet and Soulé in [12], for all noetherian schemes of finite Krull dimension. This implies that for this class of schemes, the operations defined by Grayson satisfy the usual identities for the Adams operations in a (special) λ-ring (cf. Section 4.1).

A similar conclusion is reached with the λ-operations defined by Grayson in [15]. We show that his operations agree with the ones defined by Gillet and Soulé on the algebraic K-groups of degree higher than 0. Since Grayson had already shown that his operations are the usual ones on degree 0, we conclude that they agree at all degrees. This implies that these λ-operations satisfy the usual identities of a special λ-ring as well.

The second specific application of this work is a proof that the regulator defined by Burgos and Wang in [6] is the Beilinson regulator. The proof provided here is simpler than the one given in [6], where delooping in K-theory was required.

The results of this paper are further exploited in the paper by the author [8], where an explicit chain morphism representing the Adams operations on higher algebraic K-theory with rational coefficients is constructed. Furthermore, in [4], Burgos and the author defined a morphism in the derived category of complexes from a chain complex computing higher algebraic Chow groups to Deligne–Beilinson cohomology. A slight modification of the tools developed here allows

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one to prove that this morphism induces the Beilinson regulator. This result ultimately implies that the regulator defined by Goncharov in [13] induces the Beilinson regulator as well [5].

The techniques used in this paper rely on the generalized cohomology theory of simplicial sheaves described by Gillet and Soulé in [12]. Roughly speaking, the idea is that any map from K-theory to K-theory or to a cohomology theory is characterized on higher degrees ($K_i, i \geq 1$), by its value on the virtual vector bundles of rank 0 over the simplicial classifying scheme $B.GL_N$.

More explicitly, let $\mathbf{C}$ be the big Zariski site over a noetherian finite dimensional scheme S. For any integers $N, k \geq 0$ denote by $B.GL_{N/S}$ the simplicial scheme $B.GL_N \times_{\mathbb{Z}} S$ and let $Gr_S(N, k)$ be the Grassmannian scheme over S. If $S.\mathcal{P}$ denotes the Waldhausen simplicial sheaf computing algebraic K-theory (see Section 2.2 and the comments before Corollary 4.5), for every scheme X in $\mathbf{C}$ and for all $m \geq 0$, we have $K_m(X) \cong \pi_{m+1}(S.(\mathcal{P}(X)))$. The definition of algebraic K-groups can be extended to simplicial schemes over $\mathbf{C}$ and every sheaf map (in the homotopy category of simplicial sheaves) $S.\mathcal{P} \to S.\mathcal{P}$ induces a map $K_m(Y.) \to K_m(Y.)$ for every simplicial scheme $Y.$.

Let $\mathbb{F}.$ be a simplicial sheaf over $\mathbf{C}$. The two main consequences of our uniqueness theorem are the following.

(i) *Assume that $\mathbb{F}.$ is weakly equivalent to $S.\mathcal{P}$, and let $\Phi, \Phi' : S.\mathcal{P} \to \mathbb{F}.$ be two H-space maps. Then, the morphisms*

$$\Phi, \Phi' : K_m(X) \to K_m(X), \quad m \geq 0$$

agree for all schemes X in $\mathbf{C}$, if they agree on $K_0(B.GL_{N/S})$, for all $N \geq 1$.

(ii) *Assume that $\mathbb{F}.$ is weakly equivalent to $\mathcal{K}.(\mathcal{F}^*(*))$, where $\mathcal{K}.(\cdot)$ is the sheaf version of the Dold–Puppe functor, and $\mathcal{F}^*(*)$ is a good enough (cf. Section 5.2) suitable graded sheaf giving a twisted duality cohomology theory in the sense of Gillet ([10]). Let $\Phi, \Phi' : S.\mathcal{P} \to \mathbb{F}.$ be two H-space maps. Then, the morphisms*

$$\Phi, \Phi' : K_m(X) \to H^*(X, \mathcal{F}^*(*)), \quad m \geq 0$$

agree for all schemes X in $\mathbf{C}$, if they agree on $K_0(Gr_S(N, k))$, for all N, k.

It follows from (i) that there is a unique way to extend the Adams operations from the Grothendieck group of vector bundles over a scheme X to higher K-theory by means of a sheaf map (in the homotopy category of simplicial sheaves). Analogously, the result (ii) implies that there is a unique way to extend the Chern character of vector bundles over a scheme X to higher degrees by means of a sheaf map (in the homotopy category of simplicial sheaves).

The paper is organized as follows. The first two sections are dedicated to reviewing part of the theory developed by Gillet and Soulé in [12]. More concretely, in Section 1 we recall the main concepts of the homotopy theory of simplicial sheaves and generalized cohomology theories. In Section 2 we explain how K-theory can be given in this setting. We introduce the simplicial sheaves $\mathbb{K}^N_. = \mathbb{Z} \times \mathbb{Z}_\infty B.GL_N$ and $\mathbb{K}. = \mathbb{Z} \times \mathbb{Z}_\infty B.GL$. In Section 3, we consider compatible systems of pointed maps $\Phi_N : \mathbb{K}^N_. \to \mathbb{F}.$, for $N \geq 1$ and $\mathbb{F}.$ a pointed simplicial sheaf, and provide general characterization theorems for maps from higher algebraic K-groups.

In the last three sections we develop the application of the characterization results to K-theory and to cohomology theories. Section 4 is devoted to maps from K-theory to K-theory, concretely to the Adams operations and λ-operations on higher algebraic K-theory. A characterization of these operations is given and the comparison to Grayson's Adams operations and λ-operations is provided. In Section 5, we consider maps from the K-groups to suitable sheaf cohomologies. We give a characterization of the Chern character and of the Chern classes for the higher algebraic K-groups of a scheme. We finish in Section 6 with a few words about the applications of these results in the works [8] and [4].

1. The homotopy category of simplicial sheaves

We review here the main definitions and properties of homotopy theory of simplicial sheaves. For more details about this topic see [12]. For general facts and definitions about model categories we refer the reader to for instance [19].

Let $\mathbf{C}$ be a site and let $\mathbf{T} = T(\mathbf{C})$ be the (Grothendieck) topos of sheaves on $\mathbf{C}$. We will always suppose that $\mathbf{T}$ has enough points (see [29, Section IV 6.4.1]).

Let $\mathbf{sT}$ be the category of simplicial objects in $\mathbf{T}$. One identifies $\mathbf{sT}$ with the category of sheaves of simplicial sets on $\mathbf{C}$. An object of $\mathbf{sT}$ is called a *space*.

1.1. The structure of the simplicial model category

The category $\mathbf{sT}$ is endowed with a structure of (injective) simplicial model category in the sense of Quillen [24]. This result is due to Joyal; a proof of it can be found in [22, Cor. 2.7]. Here we recall the definitions that give a simplicial model structure to $\mathbf{sT}$.

The structure of *model category* of $\mathbf{sT}$ is given as follows. Let $X.$ be a space in $\mathbf{sT}$. One defines $\pi_0(X.)$ to be the sheaf associated with the presheaf

$$U \mapsto \pi_0(X.(U)), \quad \text{for } U \in \mathrm{Obj}(\mathbf{C}).$$

Let $\mathbf{C}|U$ be the site of objects over U as described in [29, Section III 5.1], and let $\mathbf{T}|U$ denote the corresponding topos. For every object $X.$ in $\mathbf{sT}$, let $X.|U$ be the restriction of $X.$ to $\mathbf{sT}|U$. Then, for every $U \in \mathrm{Obj}(\mathbf{C})$, $x \in X_0(U)$ a vertex of the simplicial set $X_\cdot(U)$, and every integer $n > 0$, one defines $\pi_n(X_\cdot|U, x)$ to be the sheaf associated with the presheaf

$$V \mapsto \pi_n(X_\cdot(V), x), \quad \text{for } V \in \mathrm{Obj}(\mathbf{C}|U).$$

Let $X_\cdot$, $Y_\cdot$ be two spaces and let $f : X_\cdot \to Y_\cdot$ be a map.

(i) The map f is called a *weak equivalence* if the induced map $f_* : \pi_0(X_\cdot) \to \pi_0(Y_\cdot)$ is an isomorphism and, for all $n > 0$, $U \in \mathrm{Obj}(\mathbf{C})$ and $x \in X_0(U)$, the natural maps

$$f_* : \pi_n(X_\cdot|U, x) \to \pi_n(Y_\cdot|U, f(x))$$

are isomorphisms.

(ii) The map f is called a *cofibration* if for every $U \in \mathrm{Obj}(\mathbf{C})$, the induced map

$$f(U) : X_\cdot(U) \to Y_\cdot(U)$$

is a cofibration of simplicial sets, i.e. it is a monomorphism.

(iii) The map f is called a *fibration* if it has the right lifting property with respect to trivial cofibrations.

Observe that since the only map $\emptyset \to X_\cdot$ is always a monomorphism, all objects $X_\cdot$ in **sT** are cofibrant.

Let **sSets** denote the category of simplicial sets. The structure of *simplicial category* of **sT** is given by the following definitions:

(i) There is a functor **sSets** $\to$ **sT**, which sends every simplicial set $K_\cdot$ to the sheafification of the constant presheaf that takes the value $K_\cdot$ for every U in $\mathbf{C}$.

(ii) For every space $X_\cdot$ and every simplicial set $K_\cdot$, the direct product $X_\cdot \times K_\cdot$ in **sT** is the simplicial sheaf given by

$$[n] \mapsto \coprod_{\sigma \in K_n} X_n,$$

and induced face and degeneracy maps.

(iii) Let $X_\cdot$, $Y_\cdot$ be two spaces and let $\Delta^n_\cdot$ be the standard n-simplex in **sSets**. The simplicial set $\underline{\mathrm{Hom}}(X_\cdot, Y_\cdot)$ is the functor

$$[n] \mapsto \mathrm{Map}_{\mathbf{sT}}(X_\cdot \times \Delta^n_\cdot, Y_\cdot).$$

Note that by definition, a map is a cofibration of spaces if and only if it is a sectionwise cofibration of simplicial sets. For fibrations and weak equivalences this is not always true. However, it follows from the definition that a sectionwise weak equivalence is a weak equivalence of spaces.

1.2. Fibrant resolutions and the homotopy category

Let $\mathcal{C}$ be any model category and $f : X \to Y$ a map. By definition, there exist two factorizations (α, β) of f,

1. $f = \beta \circ \alpha$, with α a cofibration and β a trivial fibration,
2. $f = \delta \circ \gamma$, where γ is a trivial cofibration and δ is a fibration.

A *fibrant resolution* of an object X is a fibrant object $X^\sim$ together with a trivial cofibration $X \xrightarrow{\sigma} X^\sim$.

One can form the *homotopy category* $\mathrm{Ho}(\mathcal{C})$, associated with $\mathcal{C}$, by formally inverting the weak equivalences. For any two spaces X, Y, one denotes by $[X, Y]$ the set of maps between X and Y in this category.

When $\mathcal{C} = \mathbf{sT}$, since all spaces $X_\cdot$ in **sT** are cofibrant, if $Y_\cdot$ is a fibrant space, then

$$[X_\cdot, Y_\cdot] = \pi_0\underline{\mathrm{Hom}}(X_\cdot, Y_\cdot) = \mathrm{Map}_{\mathbf{sT}}(X, Y)/\text{homotopy equivalence},$$

that is, $[X_\cdot, Y_\cdot]$ is the set of *strictly simplicial homotopy classes* or *homotopy classes* of maps $X_\cdot \to Y_\cdot$ (cf. [19], Def. 7.3.2, Def. 9.5.2 and Prop. 9.5.24). More generally, for arbitrary $Y_\cdot$, if $Y_\cdot \to Y_\cdot^\sim$ is any fibrant resolution of $Y_\cdot$, then $[X_\cdot, Y_\cdot] = \pi_0\underline{\mathrm{Hom}}(X_\cdot, Y_\cdot^\sim)$.

Suppose that $Y_\cdot \to \hat{Y}_\cdot$ is a weak equivalence (not necessarily also a cofibration) and $\hat{Y}_\cdot$ is fibrant. Then, if $Y_\cdot^\sim$ is any fibrant resolution, there exists a weak equivalence $Y_\cdot^\sim \to \hat{Y}_\cdot$. Therefore, by [19, 9.5.12],

$$[X_\cdot, Y_\cdot] = \pi_0\underline{\mathrm{Hom}}(X_\cdot, \hat{Y}_\cdot).$$

Consider $X_\cdot, Y_\cdot \in \mathrm{Obj}(\mathbf{sT})$ and $f : X_\cdot \to Y_\cdot$ a morphism. Suppose that $Y_\cdot$ is fibrant and let $X_\cdot^\sim$ be a fibrant resolution of $X_\cdot$. Then, f factors uniquely (up to homotopy) through $X_\cdot^\sim$, i.e. there exists a map in **sT**, $f^\sim : X_\cdot^\sim \to Y_\cdot$, unique up to homotopy under $X_\cdot$, such that the following diagram is commutative:

$$\begin{tikzcd}
X_\cdot \arrow[r, "f"] \arrow[d, "\sigma"', "\sim"] & Y_\cdot \\
X_\cdot^\sim \arrow[ur, "f^\sim"'] &
\end{tikzcd}$$

See [19, 8.1.6] for a proof. Therefore, there is a map

$$[X_., Y_.] \to [X_.^{\sim}, Y_.]$$

obtained by the factorizations.

1.3. Pointed simplicial sheaves

Let $\mathbf{sT}_*$ denote the category of pointed spaces. Following [12], if $X_.$ and $Y_.$ are two pointed spaces with $Y_.$ fibrant, then one denotes by $[X_., Y_.]_0$ the set of homotopy classes of base point preserving maps $X_. \to Y_.$. For arbitrary spaces $Y_.$, the set $[X_., Y_.]_0$ is defined to be $[X_., Y_.^{\sim}]_0$ for any pointed fibrant resolution $Y_. \to Y_.^{\sim}$ of $Y_.$.

In this setting, for every pointed space $X_.$ and every simplicial set $K_.$ with K_n finite for all n, the pointed product $X_. \wedge K_.$ is the simplicial sheaf given by

$$[n] \mapsto \bigvee_{\sigma \in K_n} X_n,$$

and induced face and degeneracy maps. Then if $X_.$, $Y_.$ are two pointed spaces, the pointed simplicial set $\underline{\mathrm{Hom}}^*(X_., Y_.)$ is the functor

$$[n] \mapsto \mathrm{Map}_{\mathbf{sT}_*}(X_. \wedge \Delta_.^n, Y_.).$$

Then, if $Y_.$ is a pointed fibrant space, one has

$$[X_., Y_.]_0 = \pi_0 \underline{\mathrm{Hom}}^*(X_., Y_.).$$

Let $*$ denote the final object of $\mathbf{sT}$. If $X_.$ is a simplicial sheaf, one considers its associated pointed object to be

$$X_{.+} = X_. \sqcup * .$$

1.4. The category of simplicial presheaves

Let $\mathbf{sPre}(\mathbf{C})$ be the category of simplicial presheaves on $\mathbf{C}$, i.e. the category of functors $\mathbf{C}^{op} \to \mathbf{sSets}$. Then, one defines:

- ▷ *weak equivalences* and *cofibrations* of simplicial presheaves exactly as for simplicial sheaves, and
- ▷ *fibrations* to be the maps satisfying the right lifting property with respect to trivial cofibrations.

As shown by Jardine in [22], these definitions equip $\mathbf{sPre}(\mathbf{C})$ with a model category structure. The sheafification functor

$$\mathbf{sPre}(\mathbf{C}) \xrightarrow{(\cdot)^s} \mathbf{sT}$$

induces an equivalence between the respective homotopy categories, sending weak equivalences to weak equivalences. Moreover, the natural map $X_. \to X_.^s$ is a weak equivalence of simplicial presheaves ([22, Lemma 2.6]).

1.5. Generalized cohomology theories

Let $X_.$ be any pointed space in $\mathbf{sT}$. The *suspension* of $X_.$, $S \wedge X_.$, is defined to be the space $X_. \wedge \Delta^1 / \sim$, where $\sim$ is the equivalence relation generated by $(x, 0) \sim (x, 1)$. The *loop space functor* Ω is the right adjoint functor of S in the homotopy category.

Let $A_.$ be any space in $\mathbf{sT}_*$. For every space $X_.$, one defines the *cohomology of $X_.$ with coefficients in $A_.$* as

$$H^{-m}(X_., A_.) = [S^m \wedge X_{.+}, A_.]_0, \quad m \geq 0.$$

This is a pointed set for $m = 0$, a group for $m > 0$ and an abelian group for $m > 1$.

An infinite loop spectrum $A_.^*$ is a collection of pointed spaces $\{A_.^i\}_{i \geq 0}$, together with given weak equivalences $A_.^i \xrightarrow{\sim} \Omega A_.^{i+1}$. *The cohomology of $X_.$ with coefficients in the spectrum $A_.^*$* is defined as

$$H^{n-m}(X_., A_.^*) = [S^m \wedge X_{.+}, A_.^n]_0, \quad m, n \geq 0.$$

Due to the adjointness relation between the loop space functor and the suspension, these sets depend only on the difference $n - m$. Therefore, all of them are abelian groups.

Let $A_.$ be a pointed space and assume that there is an infinite loop spectrum $A_.^*$ with $A_.^0 = A_.$. Then the cohomology groups with coefficients in $A_.$ are also defined with positive indices, with respect to this infinite loop spectrum. By abuse of notation, when there is no source of confusion, we will write $H^m(X, A_.)$, for the generalized cohomology with positive indices, instead of writing $H^m(X, A_.^*)$.

1.6. Induced morphisms

Let $A_\cdot, B_\cdot$ be two pointed spaces. Every element $f \in [A_\cdot, B_\cdot]_0$ induces functorial maps

$$[Y_\cdot, A_\cdot]_0 \xrightarrow{f_*} [Y_\cdot, B_\cdot]_0 \quad \text{and} \quad [B_\cdot, Y_\cdot]_0 \xrightarrow{f^*} [A_\cdot, Y_\cdot]_0,$$

for every pointed space $Y_\cdot$. Therefore, there are induced maps between the generalized cohomology groups

$$H^{-*}(X_\cdot, A_\cdot) \xrightarrow{f_*} H^{-*}(X_\cdot, B_\cdot)$$

for any space $X_\cdot$.

Using simplicial resolutions, these maps can be described as follows. If $B_\cdot^\sim$ is any fibrant resolution of $B_\cdot$, then f is given by a homotopy class of pointed maps $A_\cdot \to B_\cdot^\sim$. This map factorizes, uniquely up to homotopy, through a fibrant resolution of $A_\cdot$, $A_\cdot^\sim$. Therefore there is a map $f^\sim : A_\cdot^\sim \to B_\cdot^\sim$ which induces, for every $m \geq 0$, a map

$$H^{-m}(X_\cdot, A_\cdot) = \pi_0 \underline{\mathrm{Hom}}^*(S^m \wedge X_{\cdot +}, A_\cdot^\sim) \to \pi_0 \underline{\mathrm{Hom}}^*(S^m \wedge X_{\cdot +}, B_\cdot^\sim) = H^{-m}(X_\cdot, B_\cdot).$$

The description of f^* is analogous.

1.7. The Zariski topos

By the *big Zariski site*, ZAR, we mean the category of all noetherian schemes of finite Krull dimension, equipped with the Zariski topology.

Given any scheme X, one can consider the category formed by the inclusion maps $V \to U$ with U and V open subsets of X and then define the covers of $U \subseteq X$ to be the open covers of U. This is called the *small Zariski site* of X, $\mathrm{Zar}(X)$. By the *big Zariski site* of X, $\mathrm{ZAR}(X)$, we mean the category of all schemes of finite type over X equipped with the Zariski topology. The corresponding topos is named the small or big Zariski topos (over X) respectively.

Generally, one also considers subsites of the big and small Zariski sites. For instance, the site of schemes over some base scheme S, the site of all noetherian schemes of finite Krull dimension which are also smooth over a base scheme (regular, quasi-projective or projective) is a subsite of ZAR. Similar subsites can be defined in $\mathrm{ZAR}(X)$ and $\mathrm{Zar}(X)$, depending on the properties of X.

At any of these sites, one associates with every simplicial scheme $X_\cdot$ in the underlying category $\mathbf{C}$ the simplicial sheaf represented by $X_\cdot$

$$U \mapsto \mathrm{Map}_{\mathbf{C}}(U, X_n), \qquad U \in \mathrm{Obj}\mathbf{C}, \quad \forall n.$$

If X is a scheme in $\mathbf{C}$ then its associated space is the constant simplicial sheaf represented by X. This sheaf is denoted by X as well. For any simplicial sheaf $\mathbb{F}_\cdot$ and any scheme X in $\mathbf{C}$, the equality of simplicial sets

$$\mathbb{F}_\cdot(X) = \underline{\mathrm{Hom}}(X_\cdot, \mathbb{F}_\cdot)$$

is satisfied. Generally, if $X_\cdot$ is a simplicial scheme, then one defines

$$\mathbb{F}_\cdot(X_\cdot) = \operatorname*{holim}_{\overleftarrow{n}} \mathbb{F}_\cdot(X_n),$$

where holim is the homotopy limit functor defined in [2].

2. *K*-theory as a generalized cohomology

Let $X_\cdot$ be a space such that its 0-skeleton is reduced to one point. One defines $\mathbb{Z}_\infty X_\cdot$ to be the sheaf associated with the presheaf

$$U \mapsto \mathbb{Z}_\infty X_\cdot(U),$$

the functor $\mathbb{Z}_\infty$ being the Bousfield–Kan integral completion of [2, Section I]. It comes equipped with a natural map $X_\cdot \to \mathbb{Z}_\infty X_\cdot$.

Following [12, Section 3.1], we consider $(\mathbf{T}, \mathcal{O}_{\mathbf{T}})$ a ringed topos with $\mathcal{O}_{\mathbf{T}}$ unitary and commutative. Then, for any integer $N \geq 1$, the linear group of rank N in $\mathbf{T}$, $GL_N = GL_N(\mathcal{O}_{\mathbf{T}})$, is the sheaf associated with the presheaf

$$U \mapsto GL_N(\Gamma(U, \mathcal{O}_{\mathbf{T}})).$$

Let $B_\cdot GL_N = B_\cdot GL_N(\mathcal{O}_{\mathbf{T}})$ be the classifying space of this sheaf of groups. Observe that for every $N \geq 1$, there is a natural inclusion $B_\cdot GL_N \hookrightarrow B_\cdot GL_{N+1}$. Consider the space $B_\cdot GL = \bigcup_N B_\cdot GL_N$ and the following pointed spaces:

$$\mathbb{K}_\cdot = \mathbb{Z} \times \mathbb{Z}_\infty B_\cdot GL,$$
$$\mathbb{K}_\cdot^N = \mathbb{Z} \times \mathbb{Z}_\infty B_\cdot GL_N.$$

Here, $\mathbb{Z}$ is the constant simplicial sheaf given by the constant sheaf $\mathbb{Z}$, pointed with zero. For every $N \geq 1$, the direct sum of matrices together with addition over $\mathbb{Z}$ gives a map

$$\mathbb{K}^N_{\cdot} \wedge \mathbb{K}^N_{\cdot} \to \mathbb{K}_{\cdot}.$$

These maps are compatible with the natural inclusions; thus $\mathbb{K}_{\cdot}$ is equipped with an H-group structure (see [21, Prop. B.2.6]).

2.1. *K*-theory

Following [12], for any space $X_{\cdot}$, the *stable K-theory* is defined as

$$H^{-m}(X_{\cdot}, \mathbb{K}_{\cdot}) = [S^m \wedge X_{\cdot+}, \mathbb{K}_{\cdot}]_0, \quad m \geq 0,$$

and for every $N \geq 1$, the *unstable K-theory* is defined as

$$H^{-m}(X_{\cdot}, \mathbb{K}^N_{\cdot}) = [S^m \wedge X_{\cdot+}, \mathbb{K}^N_{\cdot}]_0, \quad m \geq 0.$$

Since $\mathbb{K}_{\cdot}$ is an H-space, $H^{-m}(X_{\cdot}, \mathbb{K}_{\cdot})$ are abelian groups for all m. However, $H^{-m}(X_{\cdot}, \mathbb{K}^N_{\cdot})$ are abelian groups for all $m > 0$ and in general only pointed sets for $m = 0$.

Definition 2.1. A space $X_{\cdot}$ is *K-coherent* if the natural maps

$$\varinjlim_N H^{-m}(X_{\cdot}, \mathbb{K}^N_{\cdot}) \to H^{-m}(X_{\cdot}, \mathbb{K}_{\cdot})$$

and

$$\varinjlim_N H^{m}(X_{\cdot}, \pi_n\mathbb{K}^N_{\cdot}) \to H^{m}(X_{\cdot}, \pi_n\mathbb{K}_{\cdot})$$

are isomorphisms for all $m, n \geq 0$.

(Here $H^m(X_{\cdot}, \pi_n Y)$ are the singular cohomology groups. See [12, Section 1.2] for a discussion in this language.)

2.2. Comparison to Quillen's *K*-theory

Let $(\mathbf{T}, \mathcal{O}_{\mathbf{T}})$ be a locally ringed topos (the "locally" condition is required in Lemma 2.2 below). Recall that we defined $\mathbf{T}|U$ as the topos corresponding to the site of objects over U. For every U in $\mathbf{T}$, let $\mathcal{P}(U)$ be the category of locally free $\mathcal{O}_{\mathbf{T}|U}$-modules of finite rank, called in the sequel vector bundles. Then, the function $U \mapsto \mathcal{P}(U)$ is a functor (cf. [12, Section 3.2.1]).

Let $B_{\cdot}Q\mathcal{P}$ be the simplicial sheaf obtained by sheafification of the Quillen construction applied to every $\mathcal{P}(U)$ (see [25]). If $\Omega B_{\cdot}QP$ is the loop space of $B_{\cdot}Q\mathcal{P}$, then, by the results of [12, Section 3.2.1] and [10, Prop. 2.15], we obtain:

Lemma 2.2. *In the homotopy category of simplicial sheaves, there is a natural pointed map of spaces*

$$\mathbb{Z} \times \mathbb{Z}_\infty B_{\cdot}GL \to \Omega B_{\cdot}Q\mathcal{P}$$

which is a weak equivalence. □

Observe that this means that $\mathbb{K}_{\cdot}$ has yet another H-space structure, given by Waldhausen's pairing [31] $\Omega B_{\cdot}Q\mathcal{P} \wedge \Omega B_{\cdot}Q\mathcal{P} \to \Omega B_{\cdot}Q\mathcal{P}$. As stated in [12] Section 3.2.1, the two structures agree with each other.

It follows from the lemma that for any space $X_{\cdot}$ in $\mathbf{sT}$, there is an isomorphism

$$H^{-m}(X_{\cdot}, \mathbb{K}_{\cdot}) \cong H^{-m}(X_{\cdot}, \Omega B_{\cdot}Q\mathcal{P}), \quad m \geq 0.$$

Hence, the stable K-theory of a space can be computed using the simplicial sheaf $\Omega B_{\cdot}Q\mathcal{P}$ instead of the simplicial sheaf $\mathbb{K}_{\cdot}$.

Suppose that $\mathbf{T}$ is the category of sheaves over a category of schemes $\mathbf{C}$. Let $\mathbb{K}^{\sim}_{\cdot}$ be a fibrant resolution of $\Omega B_{\cdot}Q\mathcal{P}$. For every scheme X in $\mathbf{C}$ and $m \geq 0$, there is a natural map

$$K_m(X) = \pi_m(\Omega B_{\cdot}Q\mathcal{P}(X)) \to \pi_m(\mathbb{K}^{\sim}_{\cdot}(X)) \cong H^{-m}(X, \mathbb{K}_{\cdot}). \tag{2.3}$$

The next theorem shows that many schemes are K-coherent and that Quillen K-theory agrees with stable K-theory.

Theorem 2.4 (*[12], Proposition 5*)**.** *Suppose that X is a noetherian scheme of finite Krull dimension d and that $\mathbf{T}$ is one of the following:*

1. ZAR, *the big Zariski site of all noetherian schemes of finite Krull dimension,*
2. ZAR(X), *the big Zariski site of all schemes of finite type over X,*
3. Zar(X), *the small Zariski site of X.*

Then, viewed as a $\mathbf{T}$-space, the morphisms $K_m(X) \to H^{-m}(X, \mathbb{K}_{\cdot})$ are isomorphisms for all m. Furthermore, X is K-coherent with cohomological dimension at most d. □

Remark 2.5. Let $\mathcal{C}$ be a small category of schemes over X that contains all open subschemes of its objects. Consider the subsite $Z(X)$ of $\mathrm{ZAR}(X)$ obtained by endowing $\mathcal{C}$ with the Zariski topology. Then, the statement of the theorem will be true with $\mathbf{T} = T(Z(X))$.

For instance, if X is a regular noetherian scheme of finite Krull dimension, we could consider $Z(X)$ to be the site of all regular schemes of finite type over X. Another example would be the site of all quasi-projective schemes of finite type over a noetherian quasi-projective scheme of finite Krull dimension.

Let $S.\mathcal{P}$ denote the simplicial sheaf associated with the Waldhausen simplicial presheaf on $\mathrm{ZAR}(S)$ given by

$$X \mapsto S.(\mathcal{P}(X)).$$

There is a homotopy equivalence between Quillen's and Waldhausen's constructions.

2.3. **K**-theory of simplicial schemes

Let $\mathbf{C} = \mathrm{ZAR}$ and let $X.$ be a simplicial scheme. Then, in the Quillen context, one defines

$$K_m(X.) = \pi_{m+1}(\underset{\overleftarrow{n}}{\mathrm{holim}}\, B.Q\mathcal{P}(X_n)), \quad m \geq 0. \tag{2.6}$$

For a description of the functor holim, see [2, Section XI], for the case of simplicial sets, or see [19, Section 19], for a general treatment.

Observe that the construction of the map (2.3) can be extended to simplicial schemes. A space $X.$ is said to be *finite dimensional* if there exists an $N \geq 0$ such that $X. = \mathrm{sk}_N X.$ (where sk_N means the N-th skeleton of $X.$). Note that finite dimensional spaces are called degenerate in [12].

The next proposition is found in [12, Section 3.2.3].

Proposition 2.7. *Let $X.$ be a simplicial scheme in* ZAR*. Then, the morphism* (2.3) *gives an isomorphism $K_m(X.) \cong H^{-m}(X., \mathbb{K}.)$ for all $m \geq 0$. Moreover, if $X.$ is finite dimensional, then $X.$ is K-coherent.* $\square$

In particular, in the big Zariski site, since $B.GL_N$ is a simplicial scheme for every $N \geq 1$, we have $K_m(B.GL_N) = H^{-m}(B.GL_N, \mathbb{K}.)$. However, $B.GL_N$ is not finite dimensional.

Note that in a Zariski site over a base scheme S, the simplicial sheaf $B.GL_N$ is the simplicial scheme given by the fibred product $B.GL_{N/S} = B.GL_N \times_{\mathbb{Z}} S$.

2.4. Vector bundles over a simplicial scheme

Let $X.$ be a simplicial scheme, with face maps denoted by d_i and degeneracy maps by s_i. A *vector bundle $E.$* over $X.$ consists of a collection of vector bundles $E_n \to X_n$, $n \geq 0$, together with isomorphisms $d_i^* E_n \cong E_{n+1}$ and $s_i^* E_{n+1} \cong E_n$ for all face and degeneracy maps. Moreover, these isomorphisms should satisfy the simplicial identities. By a *morphism of vector bundles* we mean a collection of morphisms at each level, compatible with these isomorphisms. An *exact sequence* of vector bundles is an exact sequence at every level.

Let $\mathrm{Vect}(X.)$ be the exact category of vector bundles over $X.$ and consider the algebraic K-groups of $\mathrm{Vect}(X.)$, $K_m(\mathrm{Vect}(X.))$. These can be computed as the homotopy groups of the simplicial set $S.(\mathrm{Vect}(X.))$ given by the Waldhausen construction. For every simplicial scheme $X.$ and every $n \geq 0$, there is a natural simplicial map

$$S.(\mathrm{Vect}(X.)) \to S.(\mathrm{Vect}(X_n)).$$

By the definition of vector bundles over simplicial schemes, it induces a simplicial map

$$S.(\mathrm{Vect}(X.)) \to \underset{\overleftarrow{n}}{\mathrm{holim}}\, S.(\mathrm{Vect}(X_n)),$$

which induces a morphism

$$K_m(\mathrm{Vect}(X.)) \xrightarrow{\psi} K_m(X.), \quad m \geq 0 \tag{2.8}$$

(this follows from (2.6) and the homotopy equivalence between Quillen's and Waldhausen's constructions). At the zero level, $K_0(\mathrm{Vect}(X.))$ is the Grothendieck group on the category of vector bundles over $X.$.

Assume that $X.$ is a simplicial object in $\mathrm{ZAR}(S)$, with S a finite dimensional noetherian scheme. At the zero level, the morphism (2.8) above can be described as follows (see [12, Section 5]). Let $N.U$ denote the nerve of a covering U of $X.$. Let $E.^N$ denote the universal vector bundle over $B.GL_{N/S}$ and let $E.$ be a rank N vector bundle over $X.$. Then, there exists a hypercovering $p : N.U \to X.$ and a classifying map $\chi : N.U \to B.GL_{N/S}$, such that $p^*(E.) = \chi^*(E.^N)$. The induced map

$$\chi : N.U \to \{N\} \times \mathbb{Z}_\infty B.GL_N \to \mathbb{Z} \times \mathbb{Z}_\infty B.GL_N \to \mathbb{K}.$$

in $\mathrm{ZAR}(S)$ defines an element χ_+ in $H^0(N.U, \mathbb{K}.) = H^0(X., \mathbb{K}.) = K_0(X.)$, which is $\psi(E.)$. This description also shows that the morphism factorizes through the limit

$$\psi : K_0(\mathrm{Vect}(X.)) \to \underset{\overrightarrow{M}}{\lim}\, H^0(X., \mathbb{K}.^M) \to H^0(X., \mathbb{K}.).$$

3. Characterization of maps from K-theory

Our aim is to characterize functorial maps from K-theory. Since stable K-theory is expressed as a representable functor, a first approximation is obviously given by Yoneda's lemma. That is, given a pointed space $\mathbb{F}_.$ and a pointed map of spaces $\mathbb{K}_. \xrightarrow{\Phi} \mathbb{F}_.$, the induced maps

$$H^{-m}(X_., \mathbb{K}_.) \xrightarrow{\Phi_*} H^{-m}(X_., \mathbb{F}_.), \quad \forall m \geq 0,$$

are determined by the image of $id \in [\mathbb{K}_., \mathbb{K}_.]_0$ under the map $\Phi_* : H^0(\mathbb{K}_., \mathbb{K}_.) \to H^0(\mathbb{K}_., \mathbb{F}_.)$. Indeed, if $g \in H^{-m}(X_., \mathbb{K}_.) = [S^m \wedge X_{.+}, \mathbb{K}_.]_0$, there are induced morphisms

$$[\mathbb{K}_., \mathbb{K}_.]_0 \xrightarrow{g^*} [S^m \wedge X_{.+}, \mathbb{K}_.]_0 \quad \text{and} \quad [\mathbb{K}_., \mathbb{F}_.]_0 \xrightarrow{g^*} [S^m \wedge X_{.+}, \mathbb{F}_.]_0.$$

Then, $g = g^*(id)$ and

$$\Phi_*(g) = \Phi_* g^*(id) = g^* \Phi_*(id).$$

We will see that, under some favorable conditions the element id can be changed by other universal elements at the level of the simplicial scheme $B.GL_N$, for all $N \geq 1$.

3.1. Compatible systems of pointed maps and the Yoneda lemma

As in Section 2, let $(\mathbf{T}, \mathcal{O}_\mathbf{T})$ be a ringed topos and let $\mathbb{F}_.$ be a pointed fibrant space in $\mathbf{sT}$. For every $M' \geq M \geq 1$, consider the maps $e_{M,M'} \in [\mathbb{K}_.^M, \mathbb{K}_.^{M'}]_0$ and $e_{M,M'} \in [\mathbb{Z}_\infty B.GL_M, \mathbb{Z}_\infty B.GL_{M'}]_0$ induced by the natural inclusion $B.GL_M \hookrightarrow B.GL_{M'}$.

Denote by $\mathbb{B}_.^M$ either $\mathbb{K}_.^M$ or $\mathbb{Z}_\infty B.GL_M$ and by $\mathbb{B}_.$ either $\mathbb{K}_.$ or $\mathbb{Z}_\infty B.GL$.

Definition 3.1. A system of pointed maps $\Phi_M \in [\mathbb{B}_.^M, \mathbb{F}_.]_0$, $M \geq 1$, is said to be *compatible* if, for all $M' \geq M$,

$$\Phi_{M'} \circ e_{M,M'} = \Phi_M.$$

We associate with any $\Phi \in [\mathbb{B}_., \mathbb{F}_.]_0$ a compatible system of pointed maps $\{\Phi_M\}_{M\geq 1}$, given by the composition of Φ with the natural map from $\mathbb{B}_.^M$ into $\mathbb{B}_.$.

Every compatible system of pointed maps $\{\Phi_M\}_{M\geq 1}$ induces a natural transformation of functors from $\mathbf{sT}_*$ to sets:

$$\Phi(-) : \varinjlim_M [-, \mathbb{B}_.^M]_0 \to [-, \mathbb{F}_.]_0.$$

For every $N \geq 1$, we define $e_N \in \varinjlim_M [\mathbb{B}_.^N, \mathbb{B}_.^M]_0$ to be the image of $id \in [\mathbb{B}_.^N, \mathbb{B}_.^N]_0$ under the natural morphism $[\mathbb{B}_.^N, \mathbb{B}_.^N]_0 \xrightarrow{\sigma_N} \varinjlim_M [\mathbb{B}_.^N, \mathbb{B}_.^M]_0$.

We state here a variant of Yoneda's lemma for maps induced by a compatible system as above.

Lemma 3.2. *Let $\mathbb{F}_.$ be a pointed space in* $\mathbf{sT}$. *The map*

$$\left\{ \begin{array}{c} \textit{compatible systems} \\ \textit{of pointed maps} \\ \{\Phi_M\}_{M\geq 1},\ \Phi_M \in [\mathbb{B}_.^M, \mathbb{F}_.]_0 \end{array} \right\} \xrightarrow{\alpha} \left\{ \begin{array}{c} \textit{natural transformations} \\ \textit{of functors} \\ \Phi(-) : \varinjlim_M [-, \mathbb{B}_.^M]_0 \to [-, \mathbb{F}_.]_0 \end{array} \right\},$$

sending every compatible system of pointed maps to its induced natural transformation, is a bijection. In addition, one has $\Phi_N = \Phi(\mathbb{B}_.^N)(e_N)$ for all $N \geq 1$.

Proof. Without loss of generality, we can assume that $\mathbb{F}_.$ is fibrant. We prove the result by giving the explicit inverse arrow β of α. So, let

$$\Phi(-) : \varinjlim_M [-, \mathbb{B}_.^M]_0 \to [-, \mathbb{F}_.]_0$$

be a natural transformation of functors. Let $\Phi_N = \beta(\Phi)_N \in [\mathbb{B}_.^N, \mathbb{F}_.]_0$ be the image of e_N under Φ,

$$\Phi_N := \Phi(\mathbb{B}_.^N)(e_N).$$

Observe that the image of $e_{N,N'}$ under the map

$$[\mathbb{B}_.^N, \mathbb{B}_.^{N'}]_0 \xrightarrow{\sigma_{N'}} \varinjlim_M [\mathbb{B}_.^N, \mathbb{B}_.^M]_0,$$

is exactly e_N. Moreover, by hypothesis, there is a commutative diagram

$$\begin{tikzcd}
\lim_{\overrightarrow{M}}[\mathbb{B}_{\cdot}^{N'},\mathbb{B}_{\cdot}^{M}]_0 \arrow[r, "\Phi(\mathbb{B}_{\cdot}^{N'})"] \arrow[d, "e^*_{N,N'}"'] & {[\mathbb{B}_{\cdot}^{N'},\mathbb{F}_{\cdot}]_0} \arrow[d, "e^*_{N,N'}"] \\
\lim_{\overrightarrow{M}}[\mathbb{B}_{\cdot}^{N},\mathbb{B}_{\cdot}^{M}]_0 \arrow[r, "\Phi(\mathbb{B}_{\cdot}^{N})"'] & {[\mathbb{B}_{\cdot}^{N},\mathbb{F}_{\cdot}]_0}
\end{tikzcd}$$

which gives the compatibility of the system $\{\Phi_N\}_{N\geq 1}$. Therefore, the map β is defined.

Now let $X_{\cdot}$ be any space in $\mathbf{sT}_*$. In order to prove that β is a right inverse of α, we have to see that $\Phi(X_{\cdot})$ is the map induced by the just constructed system $\{\Phi_M\}_{M\geq 1}$. Let $f\in\lim_{\overrightarrow{M}}[X_{\cdot},\mathbb{B}_{\cdot}^{M}]_0$. Then, there exists an integer $N\geq 1$ and a map $g\in[X_{\cdot},\mathbb{B}_{\cdot}^{N}]_0$ such that $\sigma_N(g)=f$. By the commutative diagram

$$\begin{tikzcd}
{[\mathbb{B}_{\cdot}^{N},\mathbb{B}_{\cdot}^{N}]_0} \arrow[r, "\sigma_N"] \arrow[d, "g^*"'] & \lim_{\overrightarrow{M}}[\mathbb{B}_{\cdot}^{N},\mathbb{B}_{\cdot}^{M}]_0 \arrow[d, "g^*"] \\
{[X_{\cdot},\mathbb{B}_{\cdot}^{N}]_0} \arrow[r, "\sigma_N"'] & \lim_{\overrightarrow{M}}[X_{\cdot},\mathbb{B}_{\cdot}^{M}]_0,
\end{tikzcd}$$

we see that in fact, $f=\sigma_N(g)=g^*(e_N)$. Using the fact that Φ is a natural transformation, we have that the diagram

$$\begin{tikzcd}
\lim_{\overrightarrow{M}}[\mathbb{B}_{\cdot}^{N},\mathbb{B}_{\cdot}^{M}]_0 \arrow[r, "\Phi"] \arrow[d, "g^*"'] & {[\mathbb{B}_{\cdot}^{N},\mathbb{F}_{\cdot}]_0} \arrow[d, "g^*"] \\
\lim_{\overrightarrow{M}}[X_{\cdot},\mathbb{B}_{\cdot}^{M}]_0 \arrow[r, "\Phi"'] & {[X_{\cdot},\mathbb{F}_{\cdot}]_0}
\end{tikzcd} \tag{3.3}$$

is commutative. Hence, we obtain

$$\Phi(f)=\Phi(g^*(e_N))=g^*\Phi(e_N)=\beta(\Phi)_N\circ g=\alpha\beta(\Phi)(f),$$

as desired.

It remains to check that β is a left inverse of α. Let $\{\Phi_N\}_{N\geq 1}$ be a compatible system of pointed maps, let Φ be the associated transformation of functors obtained by α and let $\{\beta(\Phi)_N\}_{N\geq 1}$ be the system $\beta(\Phi)$. From the commutative diagram

$$\begin{tikzcd}
{[\mathbb{B}_{\cdot}^{N},\mathbb{B}_{\cdot}^{N}]_0} \arrow[r, "(\Phi_N)_*"] \arrow[d, "\sigma_N"'] & {[\mathbb{B}_{\cdot}^{N},\mathbb{F}_{\cdot}]_0} \\
\lim_{\overrightarrow{M}}[\mathbb{B}_{\cdot}^{N},\mathbb{B}_{\cdot}^{M}]_0 \arrow[ur, "\Phi"'] &
\end{tikzcd}$$

we deduce that

$$\Phi_N=(\Phi_N)_*(id)=\Phi\sigma_N(id)=\Phi(e_N)=\beta(\Phi)_N.$$

Therefore, β is the inverse of α and thus α is a bijection. □

Remark 3.4. The last lemma is not specific to our category and to our compatible system of maps. It could be directly generalized to any suitable category.

Definition 3.5. Let $\mathbb{F}_{\cdot}$ be a pointed space. For a map $f\in[X_{\cdot},\mathbb{F}_{\cdot}]$, let f_+ denote the induced map $f_+\in[X_{\cdot+},\mathbb{F}_{\cdot}]_0$.

Corollary 3.6. *Let $\mathbb{F}_{\cdot}$ be a space in $\mathbf{sT}_*$ and let $\{\Phi_M\}_{M\geq 1}$, $\{\Phi'_M\}_{M\geq 1}$ be two compatible systems of pointed maps $\Phi_M,\Phi'_M\in[\mathbb{Z}_\infty B_{\cdot}GL_M,\mathbb{F}_{\cdot}]$. Then, the induced maps*

$$\Phi,\Phi':\lim_{\overrightarrow{M}}[-,\mathbb{Z}_\infty B_{\cdot}GL_M]_0\to[-,\mathbb{F}_{\cdot}]_0$$

agree for all pointed spaces if and only if, in $[\mathbb{Z}_\infty B_{\cdot}GL_{N+},\mathbb{F}_{\cdot}]_0$, the following equation holds:

$$\Phi((e_N)_+)=\Phi'((e_N)_+),\quad \textit{for all } N\geq 1. \tag{3.7}$$

Proof. One implication is obvious. By Lemma 3.2, it is enough to see that for all N, $\Phi_N=\Phi'_N$, and this happens if and only if $\Phi(e_N)=\Phi'(e_N)$. Since the added point $+$ maps to the base point of $\mathbb{F}_{\cdot}$ by definition, the corollary is proved. □

Denote also by $e_N \in \lim_{\overrightarrow{M}}[B.GL_N, \mathbb{Z}_\infty B.GL_M]$ the map induced by $e_N \in [\mathbb{Z}_\infty B.GL_N, \mathbb{Z}_\infty B.GL_M]_0$ under the natural map $B.GL_N \to \mathbb{Z}_\infty B.GL_N$.

Corollary 3.8. *Let $\mathbb{F}.$ be an H-space in $\mathbf{sT}_*$ and let $\{\Phi_M\}_{M\geq 1}$, $\{\Phi'_M\}_{M\geq 1}$ be two compatible systems of pointed maps $\Phi_M, \Phi'_M \in [\mathbb{Z}_\infty B.GL_M, \mathbb{F}.]_0$. Then, the induced maps*

$$\Phi, \Phi' : \lim_{\overrightarrow{M}}[-, \mathbb{Z}_\infty B.GL_M]_0 \to [-, \mathbb{F}.]_0$$

agree for all pointed spaces if and only if, in $[B.GL_{N+}, \mathbb{F}.]_0$, the following equation holds:

$$\Phi((e_N)_+) = \Phi'((e_N)_+), \quad \textit{for all } N \geq 1. \tag{3.9}$$

Proof. This result is a consequence of Corollary 3.6. It follows from the fact that the natural map $[\mathbb{Z}_\infty B.GL_N, \mathbb{F}.]_0 \to [B.GL_N, \mathbb{F}.]_0$ is an isomorphism if $\mathbb{F}.$ is an H-space. □

3.2. Comparison on Zariski sites

Let S be a finite dimensional noetherian scheme. Fix $\mathbf{C}$, a Zariski subsite of ZAR(S) containing all open subschemes of its objects and the components of the simplicial scheme $B.GL_{N/S}$. Let $\mathbf{T} = T(\mathbf{C})$. To ease the notation, we write $B.GL_N = B.GL_{N/S}$ as above.

Let $\mathbb{K}^M_., \mathbb{K}.$ be as in the previous section. Let pr_1 and pr_2 be the projections onto the first and second components respectively:

$$\mathrm{pr}_1 : \mathbb{Z} \times \mathbb{Z}_\infty B.GL_M \to \mathbb{Z}, \qquad \mathrm{pr}_2 : \mathbb{Z} \times \mathbb{Z}_\infty B.GL_M \to \mathbb{Z}_\infty B.GL_M,$$

and let j_1, j_2 denote the inclusions obtained using the respective base points:

$$j_1 : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}_\infty B.GL_M, \qquad j_2 : \mathbb{Z}_\infty B.GL_M \to \mathbb{Z} \times \mathbb{Z}_\infty B.GL_M.$$

By Section 1.6, since these maps are pointed, there are induced morphisms

$$j_2^* : [\mathbb{K}^M_., \mathbb{F}.]_0 \to [\mathbb{Z}_\infty B.GL_M, \mathbb{F}.]_0, \qquad j_{2*} : [B., \mathbb{Z}_\infty B.GL_M]_0 \to [B., \mathbb{K}^M_.]_0$$

for pointed spaces $\mathbb{F}., B.$ and analogously for the other maps $\mathrm{pr}_1, \mathrm{pr}_2, j_1$.

Remark 3.10. Any compatible system of pointed maps $\{\Phi'_M\}_{M\geq 1}$, $\Phi'_M \in [\mathbb{Z}_\infty B.GL_M, \mathbb{F}.]_0$, induces naturally a compatible system of pointed maps $\Phi_M \in [\mathbb{K}^M_., \mathbb{F}.]_0$, on setting

$$\Phi_M = \mathrm{pr}_2^*(\Phi'_M),$$

with the property that $j_2^*(\Phi_M) = \Phi'_M$.

Let $X.$ be a space. Let us call an element of $K_0(X.)$ a *virtual vector bundle*. Then, a virtual vector bundle of rank 0 in $K_0(X.)$ is an element that can be represented by a compatible system of maps in the homotopy category of simplicial sheaves of the form $j_{2*}(g_M)$, with $g_M \in [X._+, \mathbb{Z}_\infty B.GL_M]_0$.

Note that the simplicial sheaf $\mathbb{Z}$ is degenerate in all degrees but 0 and recall that $H^{-m}(X., \mathbb{K}^M_.) = [S^m \wedge X._+, \mathbb{K}^M_.]_0$. If $m > 0$, the suspension $S^m \wedge X._+$ has only one element of degree zero, which has to be mapped to $(0, *)$ in any element of $[S^m \wedge X._+, \mathbb{K}^M_.]_0$. The degeneracy of $\mathbb{Z}$ for degrees higher than zero implies that all elements of $H^{-m}(X., \mathbb{K}^M_.)$ are of the form $j_{2*}(f_M)$, with $f_M \in [S^m \wedge X._+, \mathbb{Z}_\infty B.GL_M]_0$.

Let $X.$ be a K-coherent space and denote by

$$\widetilde{K}_m(X.) = \lim_{\overrightarrow{M}} H^{-m}(X., \mathbb{Z}_\infty B.GL_M)$$

the reduced algebraic K-groups. As noticed above, the map $j_{2*} : \widetilde{K}_m(X.) \to K_m(X.)$ is an isomorphism for $m > 0$ and, for $m = 0$, its image consists of the virtual vector bundles of rank 0. For any space $\mathbb{F}.$ and any map $\Phi : K_m(X.) \to H^{-m}(X., \mathbb{F}.)$, composition with j_{2*} provides a map $\widetilde{\Phi} : \widetilde{K}_m(X.) \to H^{-m}(X., \mathbb{F}.)$. If Φ is induced by a compatible system of pointed maps $\{\Phi_M\}_{M\geq 1}$, then $\widetilde{\Phi}$ is induced by the compatible system of pointed maps $\{j_2^*(\Phi_M)\}_{M\geq 1}$. Therefore, we obtain the following proposition.

Proposition 3.11. *Let $\mathbb{F}.$ be a space in $\mathbf{sT}_*$ and $\{\Phi_M\}_{M\geq 1}$, $\{\Phi'_M\}_{M\geq 1}$ be two compatible systems of pointed maps $\Phi_M, \Phi'_M \in [\mathbb{K}^M_., \mathbb{F}.]$. If $j_2^*(\Phi_M) = j_2^*(\Phi'_M)$ for all $M \geq 1$, then, for all K-coherent spaces $X.$, the induced maps*

$$\widetilde{\Phi}, \widetilde{\Phi}' : \widetilde{K}_m(X.) \to H^{-m}(X., \mathbb{F}.)$$

agree. □

For every $N \geq 1$, let

$$i_N = j_{2*}(e_N) \in \varinjlim_M [B.GL_N, \mathbb{K}^M_.]_0 = K_0(B.GL_N),$$

where $e_N \in \lim_{\overrightarrow{M}} [\mathbb{Z}_\infty B.GL_N, \mathbb{Z}_\infty B.GL_M]_0$.

A direct consequence of Corollary 3.8 and Proposition 3.11 is the following theorem.

Theorem 3.12. *Let $\mathbb{F}.$ be an H-space in $\mathbf{sT}_*$ and $\{\Phi_M\}_{M\geq 1}$, $\{\Phi'_M\}_{M\geq 1}$ be two compatible systems of pointed maps with $\Phi_M, \Phi'_M \in [\mathbb{K}^M_., \mathbb{F}.]_0$. If for all $N \geq 1$, on $H^0(B.GL_N, \mathbb{F}.)$, we have $\Phi((i_N)_+) = \Phi'((i_N)_+)$, then the induced maps*

$$\widetilde{\Phi}, \widetilde{\Phi'} : \widetilde{K}_m(X.) \to H^{-m}(X., \mathbb{F}.)$$

agree for all K-coherent spaces $X.$.

Proof. By hypothesis, we have that $\Phi((i_N)_+) = \Phi'((i_N)_+)$. Now observe that

$$\Phi((i_N)_+) = \Phi(j_{2*}(e_N)_+) = j_2^*(\Phi)((e_N)_+)$$

(and analogously for Φ'), and hence we obtain that

$$j_2^*(\Phi)((e_N)_+) = j_2^*(\Phi')((e_N)_+).$$

By Corollary 3.8, we have that $j_2^*(\Phi) = j_2^*(\Phi')$ and the statement follows from Proposition 3.11. $\square$

Corollary 3.13. *Let $\mathbb{F}.$ be an H-space in $\mathbf{sT}_*$ and let $\chi^1, \chi^2 \in [\mathbb{K}., \mathbb{F}.]_0$. If for all $N \geq 1$ we have $\chi^1((i_N)_+) = \chi^2((i_N)_+)$ on $H^0(B.GL_N, \mathbb{F}.)$, then the induced maps*

$$\widetilde{\chi^1}, \widetilde{\chi^2} : \widetilde{K}_m(X.) \to H^{-m}(X., \mathbb{F}.), \quad m \geq 0$$

agree for all finite dimensional simplicial schemes $X.$ in $\mathbf{C}$.

Proof. Let $\iota_M : \mathbb{K}^M_. \to \mathbb{K}.$ be the natural inclusion. Then, if $j_N \in [B.GL_N, \mathbb{K}.]$ is the image of i_N under the morphism

$$\varinjlim_M [B.GL_N, \mathbb{K}^M_.] \to [B.GL_N, \mathbb{K}.],$$

we have that $(j_N)_+ = (i_N)_+ \in K_0(B.GL_N)$. The statement is a direct consequence of Theorem 3.12, on considering the associated compatible system of pointed maps $\{\chi^1_M\}_{M\geq 1}$, $\{\chi^2_M\}_{M\geq 1}$ induced by the natural maps ι_M. $\square$

Remark 3.14. Note that $(i_N)_+ \in K_0(B.GL_N)$ is by definition a virtual vector bundle of rank 0. We have shown that if a map $K_0(X.) \to H^0(X., \mathbb{F}.)$ can be expressed on virtual vector bundles of rank 0 of $K_0(B.GL_N)$ by means of a pointed map in the homotopy category of simplicial sheaves $\mathbb{K}. \to \mathbb{F}.$, or by means of a compatible system of pointed maps in the homotopy category of simplicial sheaves $\{\mathbb{K}^M_. \to \mathbb{F}.\}_{M\geq 1}$, then it admits a unique extension to higher algebraic K-groups by means of a map or a compatible system of pointed maps in the homotopy category of simplicial sheaves.

Our results up to now show how to compare maps on higher algebraic K-groups when they agree on rank 0 virtual vector bundles. Comparison of the maps at the K_0 level usually requires more conditions to be satisfied. In many situations, it will be enough to compare the two maps on the constant vector bundles of each rank, thanks to the following observation.

Let $\pi_i : \mathbb{K}. \to \mathbb{K}.$, $i = 1, 2$, be given by $\pi_i = j_i \circ \mathrm{pr}_i$, with the inclusion and projection of the components of $\mathbb{K}.$ defined as above. Since the H-space structure of $\mathbb{K}.$, with operation denoted by $+$, is defined independently over the $\mathbb{Z}$ and $\mathbb{Z}_\infty B.GL$ components, there is a commutative diagram

$$\begin{tikzcd}
\mathbb{K}. \arrow[rr, "\pi_1\times\pi_2"] \arrow[dr, "id"'] & & \mathbb{K}. \wedge \mathbb{K}. \arrow[dl, "+"] \\
& \mathbb{K}. &
\end{tikzcd}$$

This means that for every $x \in K_0(X.)$, we have

$$x = \pi_{1*}(x) + \pi_{2*}(x). \tag{3.15}$$

Now, $\pi_{2*}(x)$ is a virtual vector bundle of rank 0, while $\pi_{1*}(x)$ is the rank of x, when this is defined. We have seen that agreement on $m > 0$ and on virtual vector bundles of rank 0 is equivalent to agreement of the two maps after composition by π_{2*}. Indeed, note that given a compatible system of pointed maps $f_M : \mathbb{K}^M_. \to \mathbb{F}.$, then $f_M \circ \pi_{2*} = \mathrm{pr}_2^* j_2^*(f_M)$.

For any map $f : K_0(X.) \to H^0(X., \mathbb{F}.)$, we have that

$$f(x) = f(\pi_{1*}(x) + \pi_{2*}(x)).$$

If f admits an expression as a function of $f(\pi_{1*}(x))$ and $f(\pi_{2*}(x))$, then we see that f is characterized by its value on the constant vector bundles and on the virtual vector bundles of rank 0. This will be the case, for instance, when f is a group morphism or for the λ-operations (see the next section).

Specifically, let

$$\begin{array}{rcl} B.GL_N & \xrightarrow{u_{r,N}} & \mathbb{Z} \times \mathbb{Z}_\infty B.GL_N \\ x & \mapsto & (r, *). \end{array}$$

be the homotopy class of the constant map.

Lemma 3.16. *Let $\mathbb{F}.$ be an H-space in* $\mathbf{sT}_*$ *and let $\{\Phi_M\}_{M\geq 1}$, $\{\Phi'_M\}_{M\geq 1}$ be two compatible systems of pointed maps. Consider the induced maps*

$$\Phi, \Phi' : \varinjlim_M [-, \mathbb{K}^M_.]_0 \to [-, \mathbb{F}.]_0.$$

If for all $r \in \mathbb{Z}$ and $N \geq 1$, the equality $\Phi((u_{r,N})_+) = \Phi'((u_{r,N})_+)$ holds in $H^0(B.GL_N, \mathbb{F}.)$, then $\Phi \circ \pi_{1*} = \Phi' \circ \pi_{1*}$.

Proof. We can assume that $\mathbb{F}.$ is fibrant. We know that for all r, N, the two maps $\Phi_N \circ (u_{r,N})_+$ and $\Phi'_N \circ (u_{r,N})_+$ are homotopic. Therefore, $\Phi_N \circ u_{r,N}$ and $\Phi'_N \circ u_{r,N}$ are also homotopic. Note that $\Phi_N \circ u_{r,N}$ factors through

$$u_{r,N} : \mathbb{Z}_\infty B.GL_N \to \{r\} \hookrightarrow \mathbb{Z} \xrightarrow{j_1} \mathbb{K}^N_. .$$

It follows that the maps $\Phi_N(r) : \{r\} \hookrightarrow \mathbb{Z} \xrightarrow{\Phi_N} \mathbb{F}.$ and $\Phi'_N(r) : \{r\} \hookrightarrow \mathbb{Z} \xrightarrow{\Phi'_N} \mathbb{F}.$ are homotopic. In terms of simplicial homotopy (which in this situation, by $\mathbb{F}.$ being fibrant, agrees with homotopy) we have that

$$\Phi_N(r) = \Phi'_N(r) \in \pi_0 \underline{\mathrm{Hom}}(\{r\}, \mathbb{F}.),$$

i.e., there exists a simplicial homotopy $H_r \in \underline{\mathrm{Hom}}_1(\{r\}, \mathbb{F}.)$ connecting the two elements. Since $\{r\}$ is a constant simplicial sheaf, and the same is true for the simplicial sheaf $\mathbb{Z}$, the homotopies H_r give a homotopy $H \in \underline{\mathrm{Hom}}_1(\mathbb{Z}, \mathbb{F}.)$ connecting $\Phi_N \circ j_1$ and $\Phi'_N \circ j_1$. It follows that $\Phi_N \circ \pi_1 = \Phi'_N \circ \pi_1$. $\square$

3.3. *A note on compatible systems of pointed maps*

Let $\mathbb{F}.$ be a fibrant pointed space. Let $\Phi_M \in [\mathbb{K}^M_., \mathbb{F}.]_0$, $M \geq 1$, be a compatible system of pointed maps. Since the structural morphisms $e_{M,M+1} : \mathbb{K}^M_. \to \mathbb{K}^{M+1}_.$ are cofibrations (sectionwise monomorphisms) we can use the *homotopy extension property of cofibrations* (see [19, Cor. 7.3.12]) and find iteratively starting with $M = 1$ a collection of pointed maps $f_M : \mathbb{K}^M_. \to \mathbb{F}.$ representing Φ_M in **sT** and commuting strictly, that is $f_{M+1}e_{M,M+1} = f_M$ in **sT**. Therefore, one obtains a pointed map

$$f : \mathbb{K}. \to \mathbb{F}..$$

A problem arises when trying to show that the homotopy class of the map f does not depend on the choices of $\{f_M\}_M$. In fact, there is a short exact sequence of pointed sets (see [20, Prop. 7.3.2])

$$* \to \lim{}^1 [\Sigma \mathbb{K}^N_., \mathbb{F}.]_0 \to [\mathbb{K}., \mathbb{F}.]_0 \to \lim [\mathbb{K}^N_., \mathbb{F}.]_0 \to *.$$

Hence, the dependence on the choices of f_M depends on the term $\lim^1 [\Sigma \mathbb{K}^N_., \mathbb{F}.]_0$.

4. Morphisms between *K*-groups

4.1. *Adams operations and λ-operations*

In this section we focus on the case where $\mathbb{F}. = \mathbb{K}.$. Then, the main application of statements 3.12 and 3.13 is to the Adams operations and λ-operations on higher algebraic K-theory.

Let $\mathbb{F}.$ be a pointed simplicial sheaf and $X.$ be any pointed space. Recall that for any map $f \in [X., \mathbb{F}.]$ we denote by f_+ the map induced on $X._+$ by sending the added point to the base point of $\mathbb{F}.$.

The Grothendieck group of a scheme X has a special λ-ring structure given by $\lambda^k(E) = \bigwedge^k E$, for any vector bundle E over X [16]. Adams operations are defined by the recursive formula

$$\Psi^k - \Psi^{k-1}\lambda^1 + \cdots + (-1)^{k-1}\Psi^1\lambda^{k-1} + (-1)^k k\lambda^k = 0. \tag{4.1}$$

Since exterior products are also defined for simplicial vector bundles, this definition extends to Adams operations and λ-operations on $K_0(\mathrm{Vect}(X.))$, for any simplicial scheme $X.$.

In the literature there are several definitions of the extension of the Adams operations of $K_0(X)$ to the higher algebraic K-groups. Our aim in this section is to give a criterion for their comparison.

Soulé, in [30], gives a special λ-ring structure to the higher algebraic K-groups of any noetherian regular scheme of finite Krull dimension extending the operations on K_0. Gillet and Soulé then generalize this result in [12], defining λ-operations for all K-coherent spaces in any locally ringed topos. We briefly recall this construction here.

Let $\mathbf{R}_{\mathbb{Z}}(GL_N)$ be the Grothendieck group of representations of the general linear group scheme $GL_{N/\mathbb{Z}}$. The properties of $\mathbf{R}_{\mathbb{Z}}(GL_N)$ that concern us are:

(1) $\mathbf{R}_{\mathbb{Z}}(GL_N)$ has a special λ-ring structure (cf. [28]).
(2) For any locally ringed topos, there is a ring morphism

$$\varphi : \mathbf{R}_{\mathbb{Z}}(GL_N) \to H^0(B.GL_N, \mathbb{K}.).$$

The morphism φ is defined as follows. Let $\rho : GL_N/\mathbb{Z} \to GL_M/\mathbb{Z}$ be a representation of $GL_N/\mathbb{Z}$. This induces a pointed map of simplicial sheaves $B.GL_N \to B.GL_M$. Consider the composition of this map with

$$B.GL_M \to \{M\} \times \mathbb{Z}_\infty B.GL_M \to \mathbb{K}..$$

This map gives an element $\varphi(\rho)_+ \in K_0(B.GL_N)$.

For any simplicial scheme $X.$, the operations Ψ^k_{GS}, λ^k_{GS} in $K_m(X.)$ are constructed by transferring the Adams operations and λ-operations of $\mathbf{R}_{\mathbb{Z}}(GL_N)$ to the K-theory of $B.GL_N$. Namely, consider the identity representation id_N and N times the trivial representation, denoted as N. Then there are pointed maps

$$\varphi(\Psi^k(id_N - N)),\ \varphi(\lambda^k(id_N - N)) : B.GL_N \to \mathbb{K}.$$

in the homotopy category of simplicial sheaves. Since $\mathbb{K}.$ is an H-space, there are induced maps

$$\varphi(\Psi^k(id_N - N)),\ \varphi(\lambda^k(id_N - N)) : \mathbb{Z}_\infty B.GL_N \to \mathbb{K}.$$

As we have observed (cf. Remark 3.10 and Proposition 3.11), for any simplicial scheme $X.$, this already defines Adams operations and λ-operations on the higher K-groups

$$\Psi^k_{GS}, \lambda^k_{GS} : K_m(X.) \to K_m(X.), \quad k \geq 0, m > 0,$$

and for virtual vector bundles of rank 0.

Consider the only special λ-ring structure on $\mathbb{Z}$ (cf. [1, Section 1]). On $K_0(X.)$, Adams operations can be defined by considering decomposition (3.15) and imposing additivity of Ψ^k_{GS}. In particular, adding $\varphi(\Psi^k(id_N - N))$ and the Adams operations on the $\mathbb{Z}$-component, we obtain a compatible system of pointed maps

$$\Psi^k_{GS} : \mathbb{K}^N_. \to \mathbb{K}., \quad N \geq 1,\ k \geq 0,$$

which induce the Adams operations on higher algebraic K-groups.

Addition of λ-operations on the $\mathbb{Z}$-component would not give the correct λ-operations at K_0. Instead, they should be defined so as to satisfy the λ-operation sum formula [1, Section 1]. That is, for every $x \in K_0(X.)$, let

$$\lambda^k_{GS}(x) = \sum_{i=0}^{k} \lambda^i(\pi_{1*}(x))\lambda^{k-i}_{GS}(\pi_{2*}(x)),$$

where the $\lambda^i(\pi_{1*}(x))$ are given by the special λ-ring structure of $\mathbb{Z}$. Note that for virtual vector bundles of rank 0 the definition is consistent, since $\lambda^i(0) = 0$ for $i > 0$ and $\lambda^0(0) = 1$.

Consider the product structure on $K(X.) = \bigoplus_{m\geq 0} K_m(X.)$ for which we set that $\bigoplus_{m>0} K_m(X.)$ is a square zero ideal, and is the cup product otherwise. Then, it follows from [12, Theorem 3] that these operations equip $K(X.)$ with a special λ-ring structure. This theorem is stated in [12] in a general category of simplicial sheaves. Therefore, one cannot guarantee that the degree 0 algebraic K-group has a unit and so λ^0 is not always defined. However, note that restricting ourselves to simplicial schemes, $K_0(X.)$ always has a unit and so λ^0 is defined.

In particular, we obtained that the Adams operations satisfy the identities of a special λ-ring. By this we mean that the maps

$$\Psi^k = \Psi^k_{GS} : \bigoplus_{m\geq 0} K_m(X) \to \bigoplus_{m\geq 0} K_m(X)$$

are ring morphisms for all k, $\Psi^1 = id$ and $\Psi^k \circ \Psi^l = \Psi^l \circ \Psi^k = \Psi^{kl}$ for all k, l.

A general result (see [1]) states that if R is a commutative $\mathbb{Q}$-algebra with unit endowed with morphisms $\Psi^k : R \to R$, for $k \geq 0$, satisfying the identities above, then R is a special λ-ring. The λ-operations are deduced using the formula relating the λ-operations and Adams operations (4.1).

4.2. Uniqueness theorems on Adams operations

Let S be a finite dimensional noetherian scheme. Fix $\mathbf{C}$, a Zariski subsite of ZAR(S) as in Section 3.2, and denote by $\mathbf{T}$ the corresponding topos. Let $B.GL_N = B.GL_{N/S}$.

Remark 4.2. Recall that we defined, in Section 2.4, $E_{.}^{N}$ to be the universal vector bundle over $B.GL_N$. Note that by construction, when $X. = B.GL_N$, we have $\varphi(id_N - N)_+ = \psi(E_{.}^{N} - N)$, with N the trivial vector bundle of rank N. Hence, it follows that

$$\psi(E_{.}^{N} - N) = (i_N)_+ \in K_0(B.GL_N).$$

The following theorems are a consequence of statements 3.12 and 3.13 applied to the present situation.

Since ψ is compatible with the Adams operations, and exterior products on representations correspond to exterior products on vector bundles, one has (cf. [12, Section 5])

$$\varphi(\Psi^k(id_N - N))_+ = \psi(\Psi^k(E_{.}^{N} - N)).$$

One can easily check that $\varphi(\Psi^k(id_N - N)) = \Psi_{GS}^k(i_N)$. Therefore, we have

$$\Psi_{GS}^k((i_N)_+) = \psi(\Psi^k(E_{.}^{N} - N)).$$

Theorem 4.3 (*Adams Operations*)**.** *Let $\{\rho_N\}_{N\geq 1}$, $\rho_N \in [\mathbb{K}_{.}^{N}, \mathbb{K}.]_0$, be a compatible system of pointed maps. Let ρ be the induced morphism*

$$\varinjlim_{M} H^*(-, \mathbb{K}_{.}^{M}) \to H^*(-, \mathbb{K}.).$$

If, for all $N \geq 1$ we have $\rho((i_N)_+) = \psi(\Psi^k(E_{.}^{N} - N))$, then, for all finite dimensional simplicial schemes $X.$ in **C**, *ρ agrees with $\Psi_{GS}^k : K_m(X.) \to K_m(X.)$ for $m > 0$, and for $m = 0$ on all virtual vector bundles of rank 0.*

If, in addition, ρ is additive on $K_0(X.)$ and for all $r \in \mathbb{Z}$, $\rho((u_{r,N})_+) = (u_{\psi^k(r),N})_+$, then, ρ agrees with $\Psi_{GS}^k : K_0(X.) \to K_0(X.)$.

Proof. The first part follows from Theorem 3.12. The second statement follows from Lemma 3.16. □

Since the Adams operations are group morphisms for all higher algebraic K-groups, it is natural to expect that they will be induced by pointed H-space maps

$$\mathbb{K}. \to \mathbb{K}.$$

in Ho(**sT**). The next two corollaries follow easily from the last theorem.

Corollary 4.4. *Let $\rho \in [\mathbb{K}., \mathbb{K}.]_0$ be a pointed H-space map in the homotopy category of simplicial sheaves on* **C**. *If, for all $N \geq 1$ and $r \in \mathbb{Z}$,*

- *$\rho((i_N)_+) = \psi(\Psi^k(E_N - N))$ and*
- *$\rho((u_{r,N})_+) = \psi(\Psi^k((u_{r,N})_+))$,*

then ρ agrees with the Adams operation Ψ_{GS}^k, for all finite dimensional simplicial schemes in **C**.

Proof. The result is a consequence of Corollary 3.13 and Lemma 3.16. □

Corollary 4.5. *Let $\rho \in [S.\mathcal{P}, S.\mathcal{P}]_0$ be a pointed H-space map. If for all $N \geq 1$ and some $k \geq 1$ the following square commutes:*

$$\begin{tikzcd}
K_0(\mathrm{Vect}(B.GL_{N/S})) \arrow[r, "\psi"] \arrow[d, "\Psi^k"'] & K_0(B.GL_{N/S}) \arrow[d, "\rho"] \\
K_0(\mathrm{Vect}(B.GL_{N/S})) \arrow[r, "\psi"'] & K_0(B.GL_{N/S}),
\end{tikzcd}$$

then the induced morphism $\rho : H^{-m}(X., S.\mathcal{P}) \to H^{-m}(X., S.\mathcal{P})$ agrees with the Adams operation $\Psi_{GS}^k : K_m(X.) \to K_m(X.)$, for all finite dimensional simplicial schemes $X.$ in **C**. □

Therefore, there is a unique (up to homotopy) way to extend the Adams operations from the Grothendieck group of simplicial schemes to higher K-theory of finite dimensional simplicial schemes, by means of a pointed map $S.\mathcal{P} \to S.\mathcal{P}$ in the homotopy category of simplicial sheaves. By this we mean that any two such maps inducing the usual Adams operations on the Grothendieck group of simplicial schemes will agree on the higher algebraic K-groups of finite dimensional simplicial schemes.

Grayson, in [14], defines Adams operations for the K-groups of any exact category with a suitable notion of tensor, symmetric and exterior product. The category of vector bundles over a scheme satisfies the required conditions, as goes the category of vector bundles over a simplicial scheme. For every simplicial scheme $X.$, he constructs

- two $(k-1)$-simplicial sets, $S.\tilde{G}^{(k-1)}(X.)$ and $\mathrm{Sub}_k(X.).$, whose diagonals are weakly equivalent to $S.(\mathcal{P}(X.))$, and
- a $(k-1)$-simplicial map $\mathrm{Sub}_k(X.). \xrightarrow{\Psi_G^k} S.\tilde{G}^{(k-1)}(X.)$.

Since his construction is functorial it induces a map of presheaves. Note, in addition, that we obtain a commutative diagram (via the natural morphism (2.8))

$$\begin{tikzcd} K_0(\mathrm{Vect}(B.GL_{N/S})) \arrow[r, "\psi"] \arrow[d, "\Psi_G^k"'] & K_0(B.GL_{N/S}) \arrow[d, "\Psi_G^k"] \\ K_0(\mathrm{Vect}(B.GL_{N/S})) \arrow[r, "\psi"'] & K_0(B.GL_{N/S}). \end{tikzcd}$$

Grayson's operations on $K_0(X.)$ are not defined using the recursive formula (4.1), but instead as the secondary Euler characteristic of the Koszul complex of a vector bundle. Using the splitting principle, he shows that his operations actually agree with the ones given in (4.1). In [9, Prop. 4.5, Remark 4.5], the author provides a proof of the agreement explicitly by constructing short exact sequences that relate the two definitions. That is, we show that, in the free abelian group generated by vector bundles, there is an equality $\Psi^k(E) - \Psi_G^k(E) = \sum_{i=1}^r d(s_i)$, where the s_i are short exact sequences and $d(A \to B \to C) = B - A - C$. The elements of these short exact sequences are combinations of tensor, exterior and symmetric products of vector bundles and are obtained by "breaking" a long exact sequence of finite length derived from the Koszul complex (using μ^p defined in [9, Def. 4.3]).

Therefore, without relying on the splitting principle, we have that Grayson's operations agree with the usual ones, for any exact category in which they can be defined.

Thus, on $K_0(\mathrm{Vect}(B.GL_{N/S}))$ we have $\Psi^k = \Psi_G^k$, and hence Corollary 4.5 leads to the following result.

Corollary 4.6. *Let S be a finite dimensional noetherian scheme. The Adams operations defined by Grayson in [14] agree with the Adams operations defined by Gillet and Soulé in [12], for every scheme in* $\mathrm{ZAR}(S)$. □

Therefore, Grayson's Adams operations endow $\bigoplus_{m\geq 0} K_m(X)\otimes\mathbb{Q}$ with a special λ-ring structure, for all finite dimensional noetherian schemes.

4.3. Uniqueness theorems on λ-operations

Let S be a finite dimensional noetherian scheme. Fix **C**, a Zariski subsite of $\mathrm{ZAR}(S)$ as in Section 3.2, and denote by **T** the corresponding topos.

As in the previous subsection, we have that $\lambda_{GS}^k((i_N)_+) = \psi(\lambda^k(E_.^N - N))$.

Theorem 4.7 (*Lambda Operations*)**.** *Let $\{\rho_N\}_{N\geq 1}$, $\rho_N \in [\mathbb{K}_.^N, \mathbb{K}.]_0$, be a compatible system of pointed maps. Let ρ be the induced morphism*

$$\rho : \varinjlim_M H^*(-, \mathbb{K}_.^M) \to H^*(-, \mathbb{K}.).$$

If, for all $N \geq 1$ $\rho((i_N)_+) = \psi(\lambda^k(E_.^N - N))$, then, for every finite dimensional simplicial scheme $X.$ in **C**, *ρ agrees with $\lambda_{GS}^k : K_m(X.) \to K_m(X.)$, for all $m > 0$ and for $m = 0$, at virtual vector bundles of rank* 0.

Proof. Let

$$\underline{\lambda}_{GS}^k : \mathbb{K}_.^N \to \mathbb{K}., \quad N \geq 1,\ k \geq 0,$$

be defined by $\varphi(\lambda^k(id_N - N))$ and extended trivially over the $\mathbb{Z}$-component, that is, $\underline{\lambda}_{GS}^k = \mathrm{pr}_2^*(\lambda_{GS}^k)$. Then, by Remark 3.10 together with Proposition 3.11, we have that, for all simplicial schemes $X.$, $\underline{\lambda}_{GS}^k = \lambda_{GS}^k$ on $K_m(X.)$ for $m > 0$ and at $m = 0$, for the virtual vector bundles of rank 0. The statement is then a consequence of Theorem 3.12 on considering the compatible system of pointed maps $\underline{\lambda}_{GS}^k$. □

Corollary 4.8 (*Lambda Operations*)**.** *Let $\rho \in [\mathbb{K}., \mathbb{K}.]_0$. If, for all $N \geq 1$ $\rho((i_N)_+) = \psi(\lambda^k(E_.^N - N))$, then, for every finite dimensional simplicial scheme $X.$ in* **C**, *ρ agrees with $\lambda_{GS}^k : K_m(X.) \to K_m(X.)$, for all $m > 0$ and for $m = 0$ on all virtual vector bundles of rank* 0. □

It follows that there is a way, unique up to homotopy, to extend the λ-operations on K_0, by means of a compatible system of pointed maps $\mathbb{Z}_\infty B.GL_N \to \mathbb{K}.$, in such a way that the restriction of the map to virtual vector bundles of rank zero gives the usual λ-operations. In particular, there is a unique way to extend the λ-operations on K_0, by means of a compatible system of pointed maps $\mathbb{K}. \to \mathbb{K}.$.

Grayson, in [15], defined λ-operations for the K-groups of any exact category $\mathcal{M}$ by means of a k-simplicial functorial map

$$\lambda_G^k : \mathrm{Sub}_k G\mathcal{M} \to G^k\mathcal{M},$$

with $G\mathcal{M}$ the G-construction of [11]. The two k-simplicial sets have diagonals homotopy equivalent to the loop space of the Quillen construction and hence are homotopy equivalent to $\mathbb{K}.$.

Since the constructions are functorial, they induce a map of presheaves. Grayson showed ([15, Section 8]) that his map induces the usual λ-operations on the K_0-groups of any exact category, and so in particular on the exact category of simplicial vector bundles over a simplicial scheme. Therefore, as for Grayson's Adams operations,

$$\lambda_G^k((i_N)_+) = \lambda_G^k(\psi(E_.^N - N)) = \psi(\lambda^k(E_.^N - N)),$$

and so from last corollary we obtain the following result.

Corollary 4.9. *Let S be a finite dimensional noetherian scheme. The λ-operations defined by Grayson in [15] agree with the λ-operations defined by Gillet and Soulé in [12], for every scheme in* $\mathrm{ZAR}(S)$. $\square$

5. Morphisms between K-theory and cohomology

5.1. Sheaf cohomology as a generalized cohomology theory

Fix $\mathbf{C}$ to be a subsite of the big Zariski site $\mathrm{ZAR}(S)$, as in Section 3.2, and denote by $\mathbf{T}$ the corresponding topos.

Consider the Dold–Puppe functor $\mathcal{K}.(\cdot)$ (see [7]), which associates with every cochain complex of abelian groups concentrated in non-positive degrees, G^* (or a chain complex concentrated in positive degrees), a simplicial abelian group $\mathcal{K}.(G^*)$, pointed with zero. It satisfies the property that $\pi_i(\mathcal{K}.(G^*), 0) = H^{-i}(G^*)$.

Now let G^* be an arbitrary cochain complex. Let $(\tau_{\leq n}G)[n]^*$ be the truncation at degree n of G^* followed by the translation by n. That is,

$$(\tau_{\leq n}G)[n]^i = \begin{cases} G^{i+n} & \text{if } i < 0, \\ \ker(d : G^n \to G^{n+1}) & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

For every n, one defines a simplicial abelian group by

$$\mathcal{K}.(G^*)^n := \mathcal{K}.((\tau_{\leq n}G)[n]^*).$$

The simplicial abelian groups $\mathcal{K}.(G^*)^n$ are pointed and form an infinite loop spectrum. Moreover, this construction is functorial on G^*.

Let $\mathcal{F}^*$ be a cochain complex of sheaves of abelian groups in $\mathbf{C}$, and let $\mathcal{K}.(\mathcal{F}^*)^*$ be the infinite loop spectrum obtained by applying sectionwise the construction above. For every n, $\mathcal{K}.(\mathcal{F}^*)^n$ is an H-space, since it is a simplicial sheaf of abelian groups.

Lemma 5.1 *([21, Prop. B.3.2]).* *Let $\mathcal{F}^*$ be a bounded below complex of sheaves on $\mathbf{C}$ and let X be a scheme in the underlying category. Then, for all $m \in \mathbb{Z}$,*

$$H^m(X, \mathcal{K}.(\mathcal{F}^*)^*) \cong H^m_{\mathrm{ZAR}}(X, \mathcal{F}^*). \quad \square$$

Here, the right hand side is the usual Zariski cohomology and the left hand side is the generalized cohomology of the infinite loop spectrum of the simplicial sheaf of groups $\mathcal{K}.(\mathcal{F}^*)^*$. Observe that since $\mathcal{K}.(\mathcal{F}^*)^*$ is an infinite loop space, we can consider generalized cohomology groups for all integer degrees. Thus we see that the usual Zariski cohomology can be expressed in terms of generalized sheaf cohomology using the Dold–Puppe functor.

5.2. Assumptions on the cohomology theory

Now fix a bounded below graded complex of sheaves $\mathcal{F}^*(*)$ of abelian groups, giving a twisted duality cohomology theory in the sense of Gillet [10]. Assuming an extra property for $\mathcal{F}^*(*)$, we will be able to reduce the comparison of maps at the level of the simplicial schemes $B.GL_N$ to the Grassmannian schemes.

Let $Gr_{\mathbb{Z}}(N, k)$ be the Grassmannian scheme over $\mathbb{Z}$ defined as a subscheme of $\mathbb{P}_{\mathbb{Z}}^{\binom{N}{k}-1}$ through the Plücker relations (see [18]).

Let $Gr(N, k) = Gr_{\mathbb{Z}}(N, k) \times_{\mathbb{Z}} S$ be the Grassmannian scheme over S. This is a projective scheme over S. Consider $E_{N,k}$ the rank N universal bundle of $Gr(N, k)$ and $\{U_k\} \xrightarrow{p} Gr(N, k)$ its standard trivialization. There is a classifying map of the vector bundle $E_{N,k}$, $\varphi_k : N.U_k \to B.GL_{N/S}$, satisfying $p^*(E_{N,k}) = \varphi_k^*(E_.^N)$. This map induces a map in Zariski cohomology

$$H^*_{\mathrm{ZAR}}(B.GL_N, \mathcal{F}^*(*)) \xrightarrow{\varphi_k^*} H^*_{\mathrm{ZAR}}(N.U_k, \mathcal{F}^*(*)) \cong H^*_{\mathrm{ZAR}}(Gr(N, k), \mathcal{F}^*(*)).$$

We will make the following extra assumption:

($*$) *For each m_0, there exists k_0 such that if $m \leq m_0$ and $k \geq k_0$, then*

$$H^m_{\mathrm{ZAR}}(B.GL_N, \mathcal{F}^*(*)) \xrightarrow{\varphi_k^*} H^m_{\mathrm{ZAR}}(N.U_k, \mathcal{F}^*(*))$$

is an isomorphism.

Common cohomology theories, such as Deligne–Beilinson cohomology (cf. [6, Section 5] or [4, Proof of Theorem 3.5]) and singular cohomology satisfy this assumption.

5.3. The uniqueness of characteristic classes

In [10], Gillet constructed *Chern classes* for higher K-theory starting from the universal Chern classes on $B.GL_N$. More specifically, let the $c_{j,N} \in H^{2j}_{ZAR}(B.GL_N, \mathcal{F}^*(j)), j \geq 1$, be given as in [10]. These induce a compatible system of pointed maps

$$\mathbb{Z}_\infty B.GL_N \to \mathcal{K}.(\mathcal{F}(j)[2j]^*),$$

which, extended trivially over the $\mathbb{Z}$ component of $\mathbb{K}^N_.$, that is, composing with pr_2, give a compatible system of pointed maps

$$c_{j,N} : \mathbb{K}^N_. \to \mathcal{K}.(\mathcal{F}(j)[2j]^*), \qquad j \geq 1. \tag{5.2}$$

For any space $X.$, the map induced after taking the generalized cohomology,

$$c_j : K_m(X.) \to H^{2j-m}_{\mathrm{ZAR}}(X., \mathcal{F}^*(j)), \qquad j \geq 1, m \geq 0,$$

is called the *j-th Chern class* and is a group morphism for all $m > 0$. For $m = 0$, it is straightforward to see that $c_j((i_N)_+) = c_{j,N} \in H^{2j}_{ZAR}(B.GL_N, \mathcal{F}^*(j))$. Therefore, by the functoriality of the Chern classes of vector bundles, the Chern classes obtained in this way give the usual Chern classes of vector bundles [17] when restricting to the K_0 of simplicial schemes. Let c_0 be obtained by projecting onto the $\mathbb{Z}$ factor of $\mathbb{K}.$, that is, the rank.

Using the standard formulas on the Chern classes [27], one obtains the *Chern character*

$$\mathrm{ch} : K_m(X.) \to \prod_{j\geq 0} H^{2j-m}_{\mathrm{ZAR}}(X., \mathcal{F}^*(j)) \otimes \mathbb{Q},$$

which is now a group morphism for all $m \geq 0$. It is induced by a compatible system of pointed maps

$$\mathrm{ch}_N : \mathbb{K}^N_. \to \prod_{j\geq 0} \mathcal{K}.(\mathcal{F}(j)[2j]^*) \otimes \mathbb{Q}. \tag{5.3}$$

Specifically, the map ch in the homotopy category of simplicial sheaves is obtained as follows. Consider $\overline{\mathrm{ch}}_N \in [\mathbb{Z}_\infty B.GL_N, \prod_{j\geq 0} \mathcal{K}.(\mathcal{F}(j)[2j]^*) \otimes \mathbb{Q}]$ to be given by the procedure described above for c_j, but with starting characteristic class $\mathrm{ch}_N \in \prod_{j\geq 0} H^{2j}_{ZAR}(B.GL_N, \mathcal{F}^*(j)) \otimes \mathbb{Q}$. Then, ch_N in (5.3) is given by addition of the identity map over $\mathbb{Z}$ (see [27] or [3] for a clear exposition when the base scheme S is a field).

The restriction of ch to $K_0(X)$ is the usual Chern character of a vector bundle.

We will now state the theorems analogous to Theorem 4.3, for maps from K-theory to cohomology, for cohomology theories satisfying the assumptions in Section 5.2. In order to do this, we should first understand better $c_j((i_N)_+)$ and $\mathrm{ch}((i_N)_+)$ for all j, N. This will be achieved by means of the Grassmannian schemes.

Define

$$\mathcal{F}^*(*) = \prod_{i\geq 0,\, j\in\mathbb{Z}} \mathcal{F}^i(j),$$

$$H^*_{\mathrm{ZAR}}(X., \mathcal{F}^*(*)) = \prod_{i\geq 0,\, j\in\mathbb{Z}} H^i_{\mathrm{ZAR}}(X., \mathcal{F}^*(j)).$$

Proposition 5.4. *Let $\mathcal{F}^*(*)$ be a bounded below graded complex of sheaves of abelian groups, giving a twisted duality cohomology theory in the sense of Gillet, which satisfies assumption ($*$). Let $\chi_1 = \{\chi_{1,N}\}$ and $\chi_2 = \{\chi_{2,N}\}$ be two compatible systems of pointed maps in* Ho(**sT**)*:*

$$\chi_{i,N} : \mathbb{K}^N_. \to \mathcal{K}.(\mathcal{F}^*(*)), \quad i = 1, 2.$$

For every scheme X in **C***, consider the induced maps*

$$\chi_1, \chi_2 : K_m(X) \to H^*_{\mathrm{ZAR}}(X, \mathcal{F}^*(*)).$$

Let $i = 1$ or 2. Then, $\chi_1 \circ \pi_{i*} = \chi_2 \circ \pi_{i*}$ if and only if this is the case for $X = Gr(N, k)$, for all N and k.

Proof. One implication is obvious. For the other implication, let χ_1^r, χ_2^r denote the projections of χ_1, χ_2 onto the r-th cohomology component. Fix r and let k_0 be an integer such that for every $k \geq k_0$ there is an isomorphism at the r-th level. Then, there are commutative diagrams

$$\begin{tikzcd}
\lim_{\substack{\rightarrow \\ M}} H^{-m}(B.GL_N, \mathbb{K}^M_.) \arrow[r, "\chi_1^r, \chi_2^r"] \arrow[d, "\varphi_k^*"'] & H^r_{\mathrm{ZAR}}(B.GL_N, \mathcal{F}^*(*)) \arrow[d, "\cong"', "\varphi_k^*"] \\
\lim_{\substack{\rightarrow \\ M}} H^{-m}(N.U_k, \mathbb{K}^M_.) \arrow[r, "\chi_1^r, \chi_2^r"] & H^r_{\mathrm{ZAR}}(N.U_k, \mathcal{F}^*(*)) \\
H^{-m}(Gr(N,k), \mathbb{K}_.) \arrow[r, "\chi_1^r, \chi_2^r"] \arrow[u, "p^*", "\cong"'] & H^r_{\mathrm{ZAR}}(Gr(N,k), \mathcal{F}^*(*)). \arrow[u, "\cong", "p^*"']
\end{tikzcd}$$

Let $x \in \lim_{\substack{\rightarrow \\ M}} H^{-m}(B.GL_N, \mathbb{K}^M_.)$. Then,

$$\begin{aligned} \chi_1^r(x) = \chi_2^r(x) &\Leftrightarrow (p^*)^{-1}\varphi_k^*\chi_1^r(x) = (p^*)^{-1}\varphi_k^*\chi_2^r(x) \\ &\Leftrightarrow \chi_1^r(p^*)^{-1}\varphi_k^*(x) = \chi_2^r(p^*)^{-1}\varphi_k^*(x). \end{aligned}$$

For $i = 2$, by Theorem 3.12, $\chi_1^r \circ \pi_{2*} = \chi_2^r \circ \pi_{2*}$ for all schemes X if they agree for $B.GL_N$ for all $N \geq 1$. For $i = 1$, by Lemma 3.16, $\chi_1^r \circ \pi_{1*} = \chi_2^r \circ \pi_{1*}$ if this is the case for the constant vector bundles on $B.GL_N$ for all $N \geq 1$. Therefore, since by hypothesis they agree for the Grassmannian scheme, the proposition is proved. □

The following two theorems follow from result 3.12 and 3.13, together with the preceding proposition.

Theorem 5.5 (*Chern Classes*). *Let $\mathcal{F}^*(*)$ be a bounded below graded complex of sheaves of abelian groups, giving a twisted duality cohomology theory in the sense of Gillet, which satisfies assumption $(*)$. Then, there is a unique way to extend to degrees $m > 0$ the j-th Chern classes of vector bundles by means of a compatible system of pointed maps $\{\rho_N \in [\mathbb{K}^N_., \mathcal{K}_.(\mathcal{F}(j)[2j]^*)]_0\}_{N\geq 1}$.* □

Theorem 5.6 (*Chern Character*). *Let $\mathcal{F}^*(*)$ be a bounded below graded complex of sheaves of abelian groups, giving a twisted duality cohomology theory in the sense of Gillet, which satisfies assumption $(*)$. Let*

$$\mathbb{K}_. \longrightarrow \prod_{j\in\mathbb{Z}} \mathcal{K}_.(\mathcal{F}(j)[2j]^*) \otimes \mathbb{Q}$$

be a pointed H-space map in $\mathrm{Ho}(\mathbf{sT})$. *The induced morphisms*

$$K_m(X) \to \prod_{j\in\mathbb{Z}} H^{2j-m}_{\mathrm{ZAR}}(X, \mathcal{F}^*(j)) \otimes \mathbb{Q}$$

agree with the Chern character defined by Gillet in [10] for every scheme X if and only if the induced map

$$K_0(X) \to \prod_{j\in\mathbb{Z}} H^{2j}_{\mathrm{ZAR}}(X, \mathcal{F}^*(j)) \otimes \mathbb{Q}$$

is the Chern character for $X = Gr(N, k)$, for all N, k.

Proof. Since ρ, ch are group morphisms, by decomposition (3.15) it is enough to see that $\rho \circ \pi_{i*} = \mathrm{ch} \circ \pi_{i*}$ for $i = 1, 2$. These follow from Proposition 5.4. □

Corollary 5.7. *Let $\mathcal{F}^*(*)$ be a bounded below graded complex of sheaves of abelian groups, giving a twisted duality cohomology theory in the sense of Gillet, which satisfies assumption $(*)$. Then, there is a unique way to extend the standard Chern character of vector bundles over schemes in* **C**, *by means of an H-space map*

$$\rho \in \left[\mathbb{K}_., \prod_{j\in\mathbb{Z}} \mathcal{K}_.(\mathcal{F}(j)[2j]^*) \otimes \mathbb{Q}\right]_0. \quad \square$$

Note that Gillet's construction gives an extension of the standard Chern classes and the Chern character of vector bundles by means of a map satisfying the conditions of Theorem 5.5 and Corollary 5.7 respectively. What we have actually shown is that the extension is unique in each case.

We deduce from these theorems that any simplicial sheaf map

$$S.\mathcal{P} \to \mathcal{K}_.(\mathcal{F}^*(*))\ (\otimes\mathbb{Q})$$

that induces either the Chern character or any Chern class map at the level of $K_0(X)$ induces the Chern character or the Chern class map on the higher K-groups of X.

Remark 5.8. Let **C** be the site of smooth complex varieties and let $\mathcal{D}^*(*)$ be a graded complex computing absolute Hodge cohomology. Burgos and Wang, in [6], constructed a pointed simplicial sheaf map $S.\mathcal{P} \to \mathcal{K}.(\mathcal{D}^*(*))$ which induces the Chern character on the higher algebraic K-groups of any smooth proper complex variety. A consequence of the last corollary is that their definition agrees with the *Beilinson regulator* (the Chern character for absolute Hodge cohomology; see [3]).

This is not a new result. Using other methods, Burgos and Wang already proved that the morphism that they defined was the same as the Beilinson regulator. The result is proved there by means of the bisimplicial scheme $B.P$ introduced by Schechtman in [26]. This introduced an unnecessary delooping, making the proof generalizable only to sheaf maps inducing group morphisms and introducing irrelevant ingredients to the proof.

6. Further applications

In [8] an explicit chain morphism which induces the Adams operations on rational algebraic K-theory for any regular noetherian scheme of finite Krull dimension is constructed.

Specifically, for any scheme X, we defined functorial chain morphisms

$$\Psi^k : \mathbb{Z}S_*\mathcal{P}(X) \to A_*(X)$$

with $\mathbb{Z}S_*\mathcal{P}(X)$ the chain complex associated with the simplicial set $S.(\mathcal{P}(X))$ and $A_*(X)$ a chain complex quasi-isomorphic to $\mathbb{Z}S_*\mathcal{P}(X)$ (and hence whose rational homology groups are isomorphic to the algebraic K-groups of X tensored by $\mathbb{Q}$). The ingredients of the construction are the transgression morphism [6], the chain complex of cubes for K-theory [23], the secondary Euler characteristic [14], and a few combinatorial formulas.

These morphisms induce simplicial maps

$$\Psi^k : S.(\mathcal{P}(X)) \to \mathcal{K}.(A(X)), \quad k \geq 0$$

and hence morphisms on the K-groups:

$$\Psi^k : K_m(X) \to K_m(X) \otimes \mathbb{Q}, \quad m \geq 0.$$

We showed that at the zero level, the morphisms induce the usual Adams operations on the Grothendieck group of vector bundles over any simplicial scheme $X.$. It follows from Corollary 4.5 that these morphisms agree with Gillet and Soulé's Adams operations for all regular noetherian schemes of finite Krull dimension.

The main application of our construction of Adams operations is the definition of a λ-ring structure on the rational arithmetic K-groups of an arithmetic variety X (see [9]).

Additionally, in [4], the author, together with Burgos defined a morphism in the derived category of complexes from a chain complex computing higher algebraic Chow groups to Deligne–Beilinson cohomology. In order to prove that the morphism induces the Beilinson regulator, it is not possible to directly apply the tools developed in this work, because the complex computing higher Chow groups is not functorial on a suitable big Zariski subsite of ZAR, and hence, it does not define a simplicial sheaf. However, a slight modification of the tools developed here allows one to prove that this morphism does indeed induce the Beilinson regulator.

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